Linear IFSs consisting of stochastic matrices

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AIMD (additive increase multiplicative decrease)

[1] Corless, King, Shorten, Wirth (2016)

Model of TCP: user $j = 1, ..., d$, $d \geq 2$, demands access to the internet,
$0 < \gamma_j < 1$ – coefficient of increase of j's share, $\sum_{j=1}^{d} \gamma_j = 1$,
$0 \leq \beta_j \leq 1$ – coefficient of decrease of j's share.

$$A = \begin{bmatrix} \beta_1 & \dots & 0 \\ 0 & \dots & \beta_d \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_d \end{bmatrix} \left[1 - \beta_1 \dots 1 - \beta_d\right],$$

$w_{n+1} = Aw_n$ — evolution of share at nth capacity event,
c — capacity of connection,
$w_n \in H_c = \{u \in \mathbb{R}^d : \sum_{i=1}^{d} u_i = c\}$.

AIMD matrix

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$$A = \begin{bmatrix} \beta_1 & \dots & 0 \\ 0 & \dots & \beta_d \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_d \end{bmatrix} [1 - \beta_1 \dots 1 - \beta_d],$$

$\beta_j \in [0,1]$, $\gamma_j \in (0,1)$, $\sum_{j=1}^{d} \gamma_j = 1$.

[1, Lemma 3.5 p.33]:
If $\max_{i=1,\dots,d} \beta_i < 1$, then A is Banach contractive on $\Delta_c = \{u \in \mathbb{R}^d : \sum_{i=1}^{d} u_i = c, u_i \geq 0\} \subseteq H_c$ with respect to $\|u\|_1 = \sum_{i=1}^{d} |u_i|$.

QUESTION:
What about $\#\{i = 1, .., d : \beta_i = 1\} = 1$?

Kantrowitz–Neumann criterion (2014) for contractivity

[2, Proposition 3.2] after improvement:

Let $A = (a_{ij})_{i,j \in \{1,\dots,d\}}$ be a column-stochastic matrix, i.e., $a_{ij} \in [0,1]$, $\sum_{i=1}^{d} a_{ij} = 1$, $i,j \in \{1,\dots,d\}$. Then the following are equivalent:

(i) A is a contraction w.r.t. $\|.\|_1$ on *some* c-hyperplane H_c;

(ii) A is a contraction w.r.t. $\|.\|_1$ on *every* H_c;

(iii) (positivity) $A^T \cdot A$ has positive all entries.

Kantrowitz–Neumann criterion (2014) for AIMD

Let $A = (a_{ij})_{i,j \in \{1,\dots,d\}}$ be a column-stochastic matrix, i.e., $a_{ij} \in [0,1]$, $\sum_{i=1}^{d} a_{ij} = 1$, $i,j \in \{1,\dots,d\}$. Then the following are equivalent:

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Corollary: If an AIMD matrix

$$A = \begin{bmatrix} \beta_1 & \dots & 0 \\ 0 & \dots & \beta_d \end{bmatrix} + \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_d \end{bmatrix} [1-\beta_1 \dots 1-\beta_d],$$

$\beta_j \in [0,1]$, $\gamma_j \in (0,1)$, $\sum_{j=1}^{d} \gamma_j = 1$, satisfies

$$\#\{i = 1,..,d : \beta_i = 1\} \leq 1$$

(equivalently, A has at most one trivial = unit vector column), then A is contractive on $(H_c, \|.\|_1)$.

Switched AIMD matrices $\Rightarrow$ IFS

Assume:
A_i – column-stochastic matrices,
$A_i^T \cdot A_i$ has positive all entries, $i = 1, \ldots, N$.

Then
for every $c \in \mathbb{R}$,
$\mathcal{F}_c = (H_c, f_i(u) = A_i \cdot u : i = 1, \ldots, N)$ is contractive w.r.t. $\|.\|_1$.

In particular, the above holds when each matrix A_i is an AIMD matrix with at most one trivial column (unit vector).

Switched AIMD matrices $\Rightarrow$ IFS attractor

(a) $\mathcal{F}_c$ has an attractor $\mathbb{A}_*(c)$, i.e., $\forall$ bounded $\emptyset \neq S \subseteq H_c$,

$$F^n(S) := \overline{\bigcup_{(i_1,\dots,i_n)\in\{1,\dots,N\}^n} f_{i_n} \circ \ldots \circ f_{i_1}(S)} \xrightarrow[n\to\infty]{\text{Hausdorff}} \mathbb{A}_*(c);$$

$\mathbb{A}_*(c) \equiv \mathbb{A}_*(1) \qquad \forall_{c\neq 0};$

(b) $(\mathcal{F}_c, (p_i)_{i=1}^N)$ has an attractive invariant distribution $\mu_*(c)$, i.e.,

$$M^n(\mu) := \sum_{(i_1,\dots,i_n)\in\{1,\dots,N\}^n} p_{i_n} \cdot \ldots \cdot p_{i_1} \cdot (\mu \circ f_{i_1}^{-1} \circ \ldots \circ f_{i_n}^{-1}) \xrightarrow[n\to\infty]{w*} \mu_*(c)$$

$\forall$ distrib. μ on H_c; $\sum_{i=1}^N p_i = 1,\ p_i > 0$; supp $\mu_*(c) = \mathbb{A}_*(c)$;

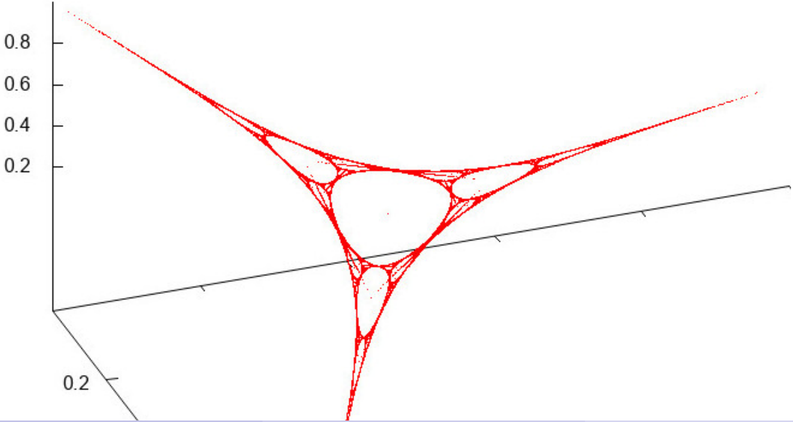
(c) (chaos game) $\mathbb{A}_*(c) = \bigcap_{n=0}^\infty \overline{\{u_m : m \geq n\}}$, where

$$\begin{cases} u_n := f_{i_n}(u_{n-1}), n \geq 1, \quad u_0 \in H_c, \\ \{1, \ldots, N\}^\infty \ni (i_n)_{n=1}^\infty \text{ – disjunctive (i.e., contains all finite words).} \end{cases}$$

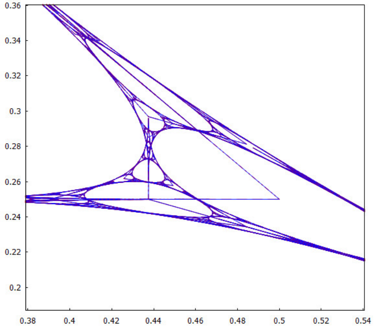
AIMD triangle

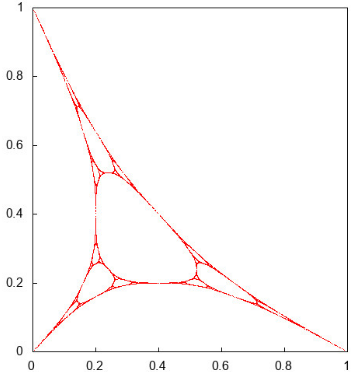
$\mathcal{F} = (\mathbb{R}^3; f_i(u) = A_i \cdot u : i = 1,2,3)$ contractive on H_c but not on $\mathbb{R}^3$.

$$A_1 = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}, A_2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}, A_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & 1 \end{pmatrix}.$$



AIMD triangle vs Kigami triangle



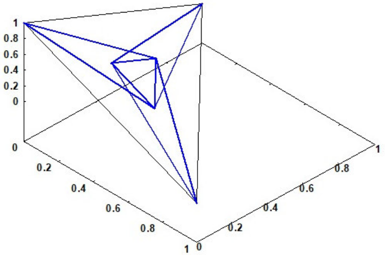


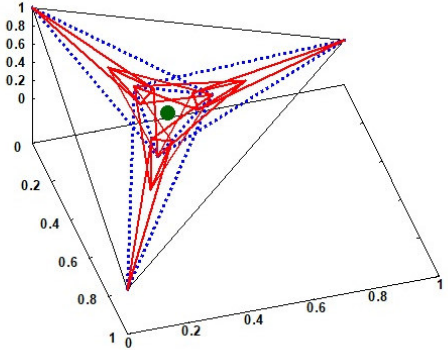
$$\mathcal{F} = (\mathbb{R}^2; f_i : i = 1, 2, 3) \qquad f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{5} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$f_2 \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{5} \cdot \left(\begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right)$$

$$f_3 \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{5} \cdot \left(\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$

AIMD triangle: analysis





$$\Delta_c = \operatorname{conv}\{(c,0,0),\ (0,c,0),\ (0,0,c)\},$$

$$\mathcal{L}_{H_c}(\mathbb{A}_*(c)) \leq \mathcal{L}_{H_c}(F^n(\Delta_c)) \leq 3^n \cdot \left(\frac{3}{16}\right)^n \mathcal{L}_{H_c}(\Delta_c) \underset{n\to\infty}{\to} 0.$$

Kantrowitz–Neumann criterion (2014) for ergodicity

[2, Theorem 3.3] & improved [2, Proposition 3.2]
Let $A = (a_{ij})_{i,j\in\{1,\ldots,d\}}$ be a column-stochastic $d \times d$-matrix. Then the following are equivalent:

(a) A is ergodic, i.e., $\exists^!_{u_*\in\Delta_1} \forall_{u\in\Delta_1} \; A^k u \underset{k\to\infty}{\longrightarrow} u_*$;

(b) $\exists_{p\geq 1} \; A^p$ contains a strictly positive row;

(c) $\exists_{p\geq 1} \; B := A^p$ is scrambling, i.e.,

$$\forall_{k,l\in\{1,\ldots,d\}} \exists_{j\in\{1,\ldots,d\}} \; b_{jk}, b_{jl} > 0;$$

(d) $\exists_{p\geq 1} \; B := A^p$ is an Edelstein contraction on Δ_1 w.r.t. $\|.\|_1$, i.e., $\|Bu - Bv\|_1 < \|u - v\|_1$ for probability vectors $u \neq v \in \Delta_1$;

(e) A is an eventual contraction on $(H_1, \|.\|_1) \supseteq \Delta_1$, i.e., $\exists_{p\geq 1} \; A^p$ is a Banach contraction on H_1;

(f) $\exists_{p\geq 1} \; (A^p)^T \cdot A^p \gg 0$.

Remark: If the above has simple proof, then the classical criterion for ergodicity of Markov matrices easily follows from metric fixed point theory.

Wielandt-type matrix

$$W_d := \begin{pmatrix} 0 & \alpha & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \dots & & \dots & \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 1-\alpha & 0 & \dots & 0 \end{pmatrix}, \alpha \in (0,1).$$

W_{10}^p	(b)= has positive row	$\gg 0$	(d)= is contractive
p	73	82	41

- [K & N & Ransford 2011] A satisfies (b) $\Rightarrow$ A^p contains positive row for $p = d^2 - 3d + 3$; optimal when $A = W_d$;
- [Horn & Johnson "Matrix Analysis"] $W_d^p \gg 0 \Leftrightarrow p \geq d^2 - 2d + 2$;
- A satisfies (d) $\Rightarrow$ A^p contractive for $p = d^2 - 3d + 3$; UNKNOWN OPTIMAL p;
- $d \leq 10 \Rightarrow p = \lfloor \frac{1}{2} \cdot (d^2 - 2d + 2) \rfloor$ is optimal for contractivity of W_d^p.

Evetually contractive IFS. Example 1

$\mathcal{F} = (\mathbb{R}^3; f_i(u) = A_i \cdot u : i = 1, 2)$, $A_i = W_3(\alpha = i/3)$,

$$A_1 = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \\ 1 & \frac{2}{3} & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \frac{2}{3} & 0 \\ 0 & 0 & 1 \\ 1 & \frac{1}{3} & 0 \end{pmatrix}.$$

$\mathcal{F}^2 = (\mathbb{R}^3; f_i \circ f_j : i, j \in \{1, 2\})$ is ℓ^1-contractive on H_c (by the KN positivity condition: $(A_i A_j)^T \cdot A_i A_j \gg 0$)

The attractor of $\mathcal{F}^2|H_c$ and $\mathcal{F}|H_c$

Evetually contractive IFS. Example 2

$\mathcal{F} = (\mathbb{R}^3; f_i(u) = A_i \cdot u : i = 1, 2)$,

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \qquad A_2 = A_1^T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

A_i are eventually contractive on H_c (KN criterion: $(A_i^2)^T \cdot A_i^2 \gg 0$), but $A_2 \cdot A_1$ is not.

$\mathcal{F}$ has no (local) attractor on H_c. (Careful verification by hand. Hint: some lines always stick out of the candidate for a local attractor).

THANK YOU

M. Corless, C. King, R. Shorten, F. Wirth, *AIMD Dynamics and Distributed Resource Allocation*, SIAM, Philadelphia, 2016.

R. Kantrowitz, M.M. Neumann, *A fixed point approach to the steady state for stochastic matrices*, Rocky Mt. J. Math. **44** (2014), no. 4, 1243-1250.