IDEAL BOUNDEDNESS OF SUBSERIES AND REARRANGEMENTS IN BANACH SPACES

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Let $\mathcal{I}$ be an admissible ideal on $\mathbb{N}$, i.e.

(1) $\emptyset \in \mathcal{I}$,
(2) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
(3) $A \subset B \wedge B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$,
(4) $\mathbb{N} \notin \mathcal{I}$,
(5) $Fin \subset \mathcal{I}$.

The simplest example of an ideal is the family of all finite subsets of $\mathbb{N}$ denoted by Fin. Consider product topology in $\{0,1\}^{\mathbb{N}} = P(\mathbb{N})$ and suppose that $\mathcal{I}$ has the Baire property ($\mathcal{I}$ is a symmetric difference of open set and nowhere dense set) in $P(\mathbb{N})$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ in a normed space is $\mathcal{I}-$bounded if there is $M > 0$ with

$$\{n \in \mathbb{N} \colon \|x_n\| > M\} \in \mathcal{I}.$$

Note that $Fin-$boundedness coincides with the notion of boundedness.

Put $S := \left\{s \in \mathbb{N}^{\mathbb{N}} \colon s \text{ is increasing}\right\}, P := \left\{p \in \mathbb{N}^{\mathbb{N}} \colon p \text{ is a bijection}\right\}$. It is easy to see that both sets are the Polish spaces, like as $\mathbb{N}^{\mathbb{N}}$ (with product topology). Denote

$$E(\mathcal{I}, (x_n)) := \left\{ s \in S \colon \left(\sum_{i=1}^{n} x_{s(i)} \right)_n \text{ is } \mathcal{I} - \text{bounded} \right\},$$

$$F(\mathcal{I}, (x_n)) := \left\{ p \in P \colon \left(\sum_{i=1}^{n} x_{p(i)} \right)_n \text{ is } \mathcal{I} - \text{bounded} \right\}$$

for a sequence $(x_n)_n$ in a normed space. Banach spaces X and Y are isomorphic if exist a linear surjection $T \colon X \to Y$ and $m, M > 0$ with $m\|x\| \le \|Tx\| \le M\|x\|$ for all $x \in X$.

Theorem 0.1. *Suppose $\mathcal{I}$ is an ideal with the Baire property, $\sum x_n$ is a series which is not unconditionally convergent in a finite-dimensional Banach space X. Then $E(\mathcal{I}, (x_n)), F(\mathcal{I}, (x_n))$ are meagre in S, P, respectively.*

Theorem 0.2. *Suppose $\mathcal{I}$ is an ideal with the Baire property and X is infinitely-dimensional Banach space X. Then X contains a copy of c_0 iff for each series $\sum x_n$ in X, which is not unconditionally convergent and $\liminf_{n\to\infty} \|x_n\| = 0$ both sets $E(\mathcal{I}, (x_n)), F(\mathcal{I}, x_n))$ are meagre in S, P, respectively.*