The Steinhaus property for regular surfaces

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It is well known that boundedness of real additive functionals on sufficiently big sets implies their continuity. In connection with this fact R. Ger and M. Kuczma defined in 1970 the following families of subsets of a real normed space X:

$$\begin{aligned}\mathfrak{B}(X) &= \{T \subset X\colon \text{ every additive functional } f : X \to \mathbb{R} \\ &\qquad \text{satisfying } \sup f(T) < \infty \text{ is continuous}\}\,, \\ \mathfrak{C}(X) &= \{T \subset X\colon \text{ every additive functional } f : X \to \mathbb{R} \\ &\qquad \text{satisfying } \sup |f(T)| < \infty \text{ is continuous}\}\,.\end{aligned}$$

It is known that nonempty open subset of X are in both of these classes. The following property connected with the famous theorem of Steinhaus is used frequently in finding "small" subsets of $\mathfrak{B}(X) \subsetneq \mathfrak{C}(X)$:

Fakt. *If* $A \subset X$ *and* $\operatorname{int}(A + A) \neq \emptyset$ *then* $A \in \mathfrak{B}(X)$.

Obviously, for the Cantor ternary set $C \subset \mathbb{R}$ we have $C + C = [0, 2]$, so $C \in \mathfrak{B}(\mathbb{R})$. There are known several other examples of "small" (nowhere dense, of Lebesgue measure zero) subsets of $\mathfrak{B}(\mathbb{R}^n)$:

(a) graphs of continuous non-affine functions (M. Kuczma 1973, WJ 1999, WJ 2005),

(b) "regular" non-flat hypersurfaces (R. Ger 1973, WJ 2019),

(c) "regular" non-flat curves (R. Ger 1973, T. Banakh, E. Jabłońska, WJ 2019).

We discuss during the talk when a regular d-dimensional surfaces $S \subset \mathbb{R}^n$ with $2 \leqslant d \leqslant n-1$ have the Steinhaus property that is $\operatorname{int}(S + S) \neq \emptyset$. We prove also that there are surfaces for which $0 \notin \operatorname{int}(S - S) \neq \emptyset$.